Reference stress solutions for plates with embedded off-set elliptical cracks under combined biaxial forces and cross-thickness bending / Soluções de estresse de referência para placas com rampas elípticas embutidas sob forças biaxiais combinadas e espessura cruzada dobrando

abstract

Reference stress method, used to J-estimation scheme, is widely applied on structural integrity assessments. A plate with embedded off-set elliptical cracks under combined biaxial forces and cross-thickness bending is considered. Reference stress solutions are derived based on the Mises yield criterion and net-section collapse method. The solutions have been compared to limit load solution of a plate with embedded rectangular crack. Then the J-integral is evaluated using the reference stress method and finite element analysis. The effect of stress parallel to the crack plane is considered. The research results show that the stress parallel to the crack plane has an effect on the J-integral of crack tip. Therefore, stress parallel to the crack plane, ignored in structural integrity assessments, may affect and lead to be non-conservative assessment results.

KEYWORDS:
□ Plate;
□ Biaxial loading;
□ Off-set elliptical cracks;
□ J-integral;
□ Reference stress method
introduction

Biaxiality usually plays a crucial role in an elastic-plastic fracture mechanics analysis and affects evaluation of the fracture resistance or crack driving force. For example, circumferential and axial cracks of pressure vessel wall are not only subjected to stress perpendicular to the crack plane, but also stress parallel to the crack plane. However, the effects of stress parallel to the crack plane on plastic collapse are ignored in existing structural integrity assessment procedures of defective component, such as R6[1] procedure. This is because the stress parallel to the crack plane has no contribution to stress intensity factor under linear elastic condition. But the stress parallel to the crack plane has an effect on limit load under elastic-plastic condition.

Recent research results show that stress parallel to the crack plane can affect limit load and J values of the defective components. For example, Yuanjun Lv et al[2] derived a global limit load solution for plates containing embedded off-set rectangular cracks under combined biaxial force/stress and through-thickness bending moment. Y lei et al[3-6] derived the limit load solutions of plates with extended surface cracks or surface cracks under combined biaxial forces and cross-thickness bending. The solutions show that...
stress parallel to the crack plane may lead to non-conservative results when used in structural integrity. Miller AG[7], X. Wang[8], Wei, L[9] and Miura N[10] also note that the effect of stress parallel to the crack plane affect structural integrity assessment results. Therefore, it is necessary to develop guidance on the treatment of the stress parallel to the defect in structural integrity assessment. Reference stress, is a very powerful tool to estimate the deformation and fracture behavior of defective components[11], has been adopted by R6 procedure. Some researchers use the reference stress method on the defective components under biaxial loading. For example, Mauro Madia[12] derived reference load solutions for plates with semi-elliptical surface cracks subjected to biaxial tensile loading. However, it is no related reports on using reference stress method on a plate with embedded off-set elliptical crack under biaxial loading.

In this paper, as a typical defect model in standard, a plate with embedded off-set elliptical crack under combined biaxial tensile/compressive force/stress and cross-thickness bending moment is reviewed. The reference stress solutions are deduced based on Mises yield criterion and net-section collapse principle. Reference stress solutions of defective component can be equivalently expressed as limit load. So the applicability of the solutions are compared to limit load solution for a plate with embedded off-set rectangular crack derived by Yuanjun Lv et al[2]. Then J-integral has been evaluated based on R6[1] procedure and compared to finite element analysis results. At last, the influence on stress parallel to the crack plane to J integral has been discussed.

section ii

reference stress expressions

2.1 Definition of geometry and load parameters

The geometry considered is an embedded elliptical crack of depth $2a$ and length $2c$ in a plate of width $2W$ and thickness $t$ (Fig. 1). The length of the plate, $2L$, is assumed to be much larger than both $2W$ and $t$. The geometric parameters, normalized crack depth, $\alpha$, normalized crack length, $\beta$, and normalized crack off-set parameter, $\kappa$, will be used in the obtained solutions and are defined as

$$\alpha = \frac{a}{t}, \beta = \frac{c}{W}, \kappa = \frac{y_{off}}{t}$$ (1)

where, $y_{off}$ is the off-set of elliptical crack from the plate symmetry plane (see Fig.1). Only cases for $\kappa \geq 0$ are considered because of the symmetry of the model.

The loads considered are an end tensile/compressive force, $N_1$, and a cross-thickness bending moment, $M_1$, are applied at the centroids of the end sections of the plate perpendicular to the crack plane; a tensile/compressive stress, $\sigma_2$, is uniformly applied at the side surfaces along the x-direction. The positive directions
of all loads are shown in Fig. 1. A yield stress $\sigma_y$ is defined for an elastic-perfectly plastic material, $N_{ll}$, $\sigma_{2L}$, and $M_{ll}$. The normalised limit loads, $n_{ll}$, $m_{ll}$, and $n_{L2}$, are defined as

$$n_{ll} = \frac{N_{ll}}{2Wt\sigma_y}, \quad m_{ll} = \frac{2M_{ll}}{Wt^2\sigma_y}, \quad n_{L2} = \frac{\sigma_{2L}}{\sigma_y} \tag{2}$$

For proportional loading, dimensionless load ratios $\lambda_1$, $\lambda_2$, and $\lambda_3$, are defined as

$$\lambda_1 = \frac{M_{ll}}{tN_1} = \frac{m_{ll}}{tN_{ll}} = \frac{n_{ll}}{4n_{ll}} = \frac{\sigma_b}{6\sigma_m} \tag{3}$$

$$\lambda_2 = \frac{\sigma_2}{\sigma_m} = \frac{n_{L2}}{n_{ll}} \tag{4}$$

$$\lambda_3 = \frac{\sigma_2}{\sigma_b} = \frac{2n_{L2}}{3m_{ll}} = \frac{\lambda_2}{6\lambda_1} \tag{5}$$

where, $\sigma_m = \frac{N_1}{2Wt}$, $\sigma_b = \frac{3M_1}{Wt^2}$

(6)

Basic assumptions and crack type classification are the same as the reference [2] described. Three types (Crack I, Crack II and Crack III) are existed in assumed stress distribution A or assumed stress distribution B. Each point at the crack ligament surface at limit state is determined by using Von Mises yield criterion. The according yield equations can meet relationship as follows after they have the normalized treatment:

$$S_i^+ = \frac{1}{2}(n_{L2} + \sqrt{4 - 3n_{L2}^2})$$

$$S_i^- = \frac{1}{2}(n_{L2} - \sqrt{4 - 3n_{L2}^2}) \tag{7}$$

where, $S_i^+ = \frac{\sigma_{i}^+}{\sigma_y}, S_i^- = \frac{\sigma_{i}^-}{\sigma_y} \tag{8}$

The distance between the neutral axis and the front surface of the plate has been defined by $\overline{y}$ (see Fig. 1). The normalised form of $\overline{y}$ is defined as

$$\delta = \frac{\overline{y}}{t} \tag{9}$$

2.2 Reference stress expressions of assumed stress distribution A

Assumed stress distribution of $z$-direction ($\sigma_i^+, \sigma_i^-$) based on lower bound theorem must meet with external load equilibrium at limit state.

1. Cases for “crack type I”

The valid region of “crack type I” meets $\delta \in \left[\frac{1}{2} - \kappa + \alpha, 1\right]$. Referring to Fig. 1 and taking force equilibrium and moment equilibrium, the following equation can be obtained.

$$\begin{cases} N_{ll} = (2W\overline{y} - \pi ac)\sigma_i^+ + 2W(t - \overline{y})\sigma_i^- \\ M_{ll} = [W\overline{y}(t - \overline{y}) - \pi acy_{off}]\sigma_i^+ - W(t - \overline{y})\overline{y}\sigma_i^- \end{cases} \tag{10}$$

Using Eqns. (1), (2), (8) and (9), Eqn. (12) can be obtained and expressed as

$$\begin{cases} n_{ll} = \left(\delta - \frac{\pi}{2}\alpha\beta\right)S_i^+ + (1 - \delta)S_i^- \\ m_{ll} = 2\delta(1 - \delta)(S_i^+ - S_i^-) - 2\pi\alpha\beta\kappa S_i^+ \end{cases} \tag{11}$$

Using Eqns. (7) and (11), Eqn. (12) can be obtained:

$$A(4 - 3n_{L2}^2) + (m_{ll} + \pi\alpha\beta n_{ll} + Bn_{L2})\sqrt{4 - 3n_{L2}^2} + 2(Cn_{L2} + n_{ll})^2 = 0 \tag{12}$$

where, the coefficients $A$, $B$, $C$ in Eqn.(12) can be determined as follows:

$$\begin{cases} A = \frac{\pi^2}{8}\alpha^2\beta^2 + \pi\alpha\beta\kappa - \frac{1}{2} \\ B = \frac{\pi^2}{4}\alpha^2\beta^2 - \frac{1}{2}(1 - 2\kappa)\pi\alpha\beta \\ C = \frac{\pi}{4}\alpha\beta - \frac{1}{2} \end{cases} \tag{13}$$

Using Eqn. (2), Eqn. (14) can be obtained:
\[ A(4\sigma_y^2 - 3\sigma_{2L}^2) + \left(\frac{2M_{el}}{W_L} + \pi\alpha\beta\frac{N_{el}}{2W_L} + B\sigma_{2L}\right)\sqrt{4\sigma_y^2 - 3\sigma_{2L}^2} + 2\left(C\sigma_{2L} + \frac{N_{el}}{2W_L}\right)^2 = 0 \]  

(14)

It can be proved that \( A < 0 \). Solving Eqn. (14) for \( \sqrt{4\sigma_y^2 - 3\sigma_{2L}^2} \) and re-arranging the result as Eqn. (15). Note that \( \sqrt{4\sigma_y^2 - 3\sigma_{2L}^2} > 0 \).

\[ \sigma_y^2 = \frac{3}{4}\sigma_{2L}^2 + \frac{\left(\frac{2M_{el}}{W_L} + \pi\alpha\beta\frac{N_{el}}{2W_L} + B\sigma_{2L}\right)^2}{4A} \]  

(15)

For the plastic collapse condition, it meets \( \sigma_{ref} = \sigma_y \). For proportional loading, the ratios between load components are constant and the reference stress can be calculated in the same way, so Eqn. (16) can be obtained:

\[ \sigma_{ref}^2 = \frac{3}{4}\sigma_y^2 + \frac{\left(\frac{2M_{el}}{W_L} + \pi\alpha\beta\frac{N_{el}}{2W_L} + B\sigma_y\right)^2}{4A} \]  

(16)

2. Cases for “crack type II”

The valid region of “crack type II” meets \( \delta \in \left[\frac{1}{2} - \kappa - \alpha, \frac{1}{2} - \kappa + \alpha\right] \), the equilibrium equations can be obtained as follows.

\[ N_{el} = (2W\bar{y} + A_e)(\sigma_y' - \sigma_y) - \pi a c \sigma_y' + 2W\sigma_y' \]
\[ M_{el} = [W\bar{y}(t - \bar{y}) - A_e(y^* - y_{eff})](\sigma_y' - \sigma_y) - \pi a c y_{eff} \sigma_y' \]  

(17)

where, \( A_e \) is the area of the part of the elliptical crack in the region with \( \sigma_y' \), \( y^* \) is the distance between the centroid of \( A_e \) and symmetry axis of elliptical crack.

Using Eqns. (1), (2), (8) and (9), Eqn. (17) can be simplified and expressed as

\[ n_{el} = \frac{\delta + \frac{1}{2}\alpha\beta f_1\left(\frac{\varepsilon}{\alpha}\right)}{\frac{\pi}{2}\alpha c S_y'} + S_y' \]
\[ m_{el} = 2\left(\delta(1 - \delta) - \alpha\beta f_1\left(\frac{\varepsilon}{\alpha}\right)\left(\frac{2}{3}\alpha f_2\left(\frac{\varepsilon}{\alpha}\right) - \kappa\right)\right)(S_y' - S_y') - 2\pi a b c S_y' \]  

(18)

where

\[ A_e = a c f_1\left(\frac{\varepsilon}{\alpha}\right) \]  

(19)
\[ y^* = \frac{2}{3}\alpha f_2\left(\frac{\varepsilon}{\alpha}\right) \]
\[ \varepsilon = \delta + \kappa - \frac{1}{2} \]
\[ f_1\left(\frac{\varepsilon}{\alpha}\right) = \arccos\frac{\varepsilon}{\alpha} - \frac{\varepsilon}{\alpha}\sqrt{1 - \left(\frac{\varepsilon}{\alpha}\right)^2} \]  

(22)
\[ f_2\left(\frac{\varepsilon}{\alpha}\right) = \left[1 - \left(\frac{\varepsilon}{\alpha}\right)^2\right]^{\frac{3}{2}} \]  

(23)

\[ f_1\left(\frac{\varepsilon}{\alpha}\right) = \frac{1}{f_1\left(\frac{\varepsilon}{\alpha}\right)} \]
In order to facilitate the value of \( f_1 \left( \frac{\varepsilon}{\alpha} \right) \), polynomial fitting formula can be gotten as follow:

\[
egin{align*}
 f_1 \left( \frac{\varepsilon}{\alpha} \right) & \approx 0.4317 \left( \frac{\varepsilon}{\alpha} \right)^3 - 2.0258 \left( \frac{\varepsilon}{\alpha} \right) + 1.5708 \\
 f_1 \left( \frac{\varepsilon}{\alpha} \right) \cdot f_2 \left( \frac{\varepsilon}{\alpha} \right) & \approx 0.5481 \left( \frac{\varepsilon}{\alpha} \right)^4 - 1.5735 \left( \frac{\varepsilon}{\alpha} \right)^2 + 1.004
\end{align*}
\]

(24)

Using Eqns. (3), (7), (19), (24), the solution of \( \alpha \) can be gotten from Eqn. (25):

\[
A_1 \left( \frac{\varepsilon}{\alpha} \right)^4 + A_2 \left( \frac{\varepsilon}{\alpha} \right)^3 + A_3 \left( \frac{\varepsilon}{\alpha} \right)^2 + A_4 \left( \frac{\varepsilon}{\alpha} \right) + A_5 = 0
\]

where,

\[
A_1 = 0.7308Z \\
A_2 = 0.21585 - 0.8634 \frac{K}{\alpha} Z \\
A_3 = \frac{2}{\beta} - 2.098 Z \\
A_4 = 1 - 1.0129 + 4.0516 \frac{K}{\alpha} - 4 \frac{K}{\alpha} \frac{K}{\alpha} Z \\
A_5 = - \frac{K}{\alpha} \beta + \left( 1.3387 - \frac{1 - 4K^2}{2\alpha^2 \beta} \right) Z
\]

(26)

where, \( Z = \frac{1 + \frac{1}{2} \left( \frac{\pi}{2} \frac{\alpha}{\beta} - 1 \right) \lambda_2}{\pi \alpha \beta \kappa \lambda_2 + 4 \lambda_1} \). Note that \(-1 \leq \frac{\varepsilon}{\alpha} \leq 1\).

Using Eqns. (7) and (18)~(23), Eqn. (27) can be obtained:

\[
A \left( 4\sigma_y^2 - 3\sigma_{z1}^2 \right) + \left[ \frac{2M_L}{W_1} \left( \frac{\varepsilon}{\alpha} \right) + \frac{\pi f_{1} \left( \frac{\varepsilon}{\alpha} \right)}{2} \right] \frac{\alpha \beta N_{W1}}{W_1} + B \sigma_{z1} = \sqrt{4\sigma_y^2 - 3\sigma_{z1}^2} + 2 \left( C \sigma_{z1} + \frac{N_{W1}}{2W_1} \right)^2 = 0
\]

(27)

where, the coefficients \( A, B, C \) in Eqn.(27) can be determined as follows:

\[
A = 2 \left( \frac{\pi}{4} - \frac{1}{2} f_1 \left( \frac{\varepsilon}{\alpha} \right) \right) \alpha^2 \beta^2 + 2 \alpha \beta f_1 \left( \frac{\varepsilon}{\alpha} \right) \left( \frac{2}{3} \alpha f_1 \left( \frac{\varepsilon}{\alpha} \right) - K \right) + \pi \alpha \beta - \frac{1}{2}
\]

\[
B = \left( \frac{\pi - 2 f_1 \left( \frac{\varepsilon}{\alpha} \right)}{\pi} \right) \left( \frac{\alpha}{\alpha} + \frac{1}{2} \right) \alpha \beta + \pi \alpha \beta
\]

\[
C = \frac{\pi}{4} \alpha \beta - \frac{1}{2}
\]

(28)

It can be proved that \( A < 0 \). Eqn. (29) can be obtained in the same method.

\[
\sigma_{ref}^2 = \frac{3}{4} \sigma_1^2 + \left[ \frac{2M_L}{W_1} \left( \frac{\varepsilon}{\alpha} \right) + \frac{f_{1} \left( \frac{\varepsilon}{\alpha} \right)}{2} \right] \frac{\alpha \beta N_{W1}}{W_1} + B \sigma_{z1} + \sqrt{4\sigma_y^2 - 3\sigma_{z1}^2} + 2 \left( C \sigma_{z1} + \frac{N_{W1}}{2W_1} \right)^2 - 8A \left( C \sigma_{z1} + \frac{N_{W1}}{2W_1} \right)^2 - 4A
\]

(29)

3 Cases for “crack type III”

The valid region of “crack type III” meets \( \delta \in \left[ 0, \frac{1}{2} - \kappa - \alpha \right] \), the equilibrium equations can be obtained.

\[
\begin{align*}
N_{IL} &= 2W \overline{T} \left( \sigma_i^1 - \sigma_i \right) + \left( 2W_t - \alpha \beta \right) \sigma_i \\
M_{IL} &= W \left( 1 - \overline{T} \right) \overline{\sigma}_i \left( \sigma_i^1 - \sigma_i \right) - \alpha \beta \sigma_{ref} \sigma_i
\end{align*}
\]

(30)

Using Eqns. (3), (4), (5), (10) and (14), Eqn. (31) can be obtained:
\[ A(4\sigma_y^2 - 3\sigma_{2z}^2) + \left( \frac{2M_{\parallel}}{W_t} - \pi\alpha\beta\frac{N_{\parallel}}{2W_t} - B\sigma_z \right) \sqrt{4\sigma_y^2 - 3\sigma_{2z}^2} + 2\left(C\sigma_{2z} + \frac{N_{\parallel}}{2W_t}\right)^2 = 0 \]

where, the coefficients \( A, B, C \) in Eqn.(32) can be determined as follows:

\[
\begin{align*}
A &= \frac{\pi^2}{8} \alpha^2 \beta^2 - \pi\alpha\beta \kappa - \frac{1}{2} \\
B &= \frac{\pi^2}{4} \alpha^2 \beta^2 - \frac{1}{2}(1 + 2\kappa) \pi\alpha\beta \\
C &= \frac{\pi}{4} \alpha\beta - \frac{1}{2}
\end{align*}
\]

It can be proved that \( A < 0 \). Eqn. (34) can be obtained in the same method.

\[
\sigma_{rref}^2 = \frac{3}{4} \sigma_z^2 + \left[ \left( \frac{2M_{\parallel}}{W_t^2} - \pi\alpha\beta\frac{N_{\parallel}}{2W_t} - B\sigma_z \right) + \sqrt{\frac{2M_{\parallel}}{W_t^2} - \pi\alpha\beta\frac{N_{\parallel}}{2W_t} - B\sigma_z} \right]^2 - 8A \left(C\sigma_{\parallel} + \frac{N_{\parallel}}{2W_t}\right)^2
\]

2.3 Reference stress expressions of assumed stress distribution B

1. Cases for “crack type I”

The valid region of “crack type I” meets \( \delta \in \left[\frac{1}{2} - \kappa + \alpha, 1\right] \), the normalized equilibrium equation can be obtained.

\[
\begin{align*}
n_{\parallel} &= (1 - \delta)S_i^+ + \left( \delta - \frac{\pi}{2} \alpha\beta \right) S_i^- \\
m_{\parallel} &= -2\delta(1 - \delta)(S_i^+ - S_i^-) - 2\pi\alpha\beta\kappa S_i^+
\end{align*}
\]

Using Eqns.(2), (3), (4), (5), (10) and (14), Eqn. (36) can be obtained:

\[ A(4\sigma_y^2 - 3\sigma_{2z}^2) + \left( \frac{2M_{\parallel}}{W_t^2} + \pi\alpha\beta\frac{N_{\parallel}}{2W_t} + B\sigma_z \right) \sqrt{4\sigma_y^2 - 3\sigma_{2z}^2} + 2\left(C\sigma_{2z} + \frac{N_{\parallel}}{2W_t}\right)^2 = 0 \]

where, the coefficients \( A, B, C \) in Eqn.(36) are the same as those defined by Eqn.(13).

Similarly, the solution can be gotten as follow.

\[
\sigma_{rref}^2 = \frac{3}{4} \sigma_z^2 + \left[ \left( \frac{2M_{\parallel}}{W_t^2} + \pi\alpha\beta\frac{N_{\parallel}}{2W_t} + B\sigma_z \right) + \sqrt{\frac{2M_{\parallel}}{W_t^2} + \pi\alpha\beta\frac{N_{\parallel}}{2W_t} - B\sigma_z} \right]^2 - 8A \left(C\sigma_{\parallel} + \frac{N_{\parallel}}{2W_t}\right)^2
\]

2. Cases for “crack type II”

The valid region of “crack type I” meets \( \delta \in \left[\frac{1}{2} - \kappa - \alpha, \frac{1}{2} - \kappa + \alpha\right] \), the normalized equilibrium equation can be obtained.
Reference stress solutions for plates with embedded off-set elliptical cracks - Yuanjun Lv et al.

\[
\begin{align*}
n_{nt} &= - \left( \delta + \frac{1}{2} \alpha \beta f_1 \left( \frac{\varepsilon}{\alpha} \right) \right) \left( S_i^* - S_i \right) - \frac{\pi}{2} \alpha \beta S_i^* + S_i^* \\
m_{nt} &= -2 \left[ \delta(1-\delta) - \alpha \beta f_1 \left( \frac{\varepsilon}{\alpha} \right) \left( \frac{2}{3} \alpha f_2 \left( \frac{\varepsilon}{\alpha} \right) - \kappa \right) \right] \left( S_i^* - S_i \right) - 2\pi\alpha\beta\kappa S_i^*
\end{align*}
\]  

(38)

Using Eqns. (2), (7) and (18)~(23), Eqn. (39) can be obtained:

\[
A \left( 4\sigma_y^2 - 3\sigma_{2L}^2 \right) - \left[ \frac{2M_{nt}}{Wt^2} + \left( \frac{\pi}{2} - f_1 \left( \frac{\varepsilon}{\alpha} \right) \right) \alpha \beta \frac{N_{nt}}{Wt} + B\sigma_{2L} \right] \sqrt{4\sigma_y^2 - 3\sigma_{2L}^2} + 2 \left( C\sigma_{2L} + \frac{N_{nt}}{2Wt} \right)^2 = 0
\]

(39)

where, the coefficients \( A, B, C \) in Eqn.(39) are the same as those defined by Eqn.(28).

Similarly, the solution can be gotten as follow.

\[
\sigma_{ref}^2 = \frac{3}{4} \sigma_y^2 + \left[ - \frac{2M_1}{Wt^2} + \left( \frac{\pi}{2} - f_1 \left( \frac{\varepsilon}{\alpha} \right) \right) \alpha \beta \frac{N_1}{Wt} + B\sigma_2 \right] \sqrt{\frac{2M_1}{Wt^2} + \left( \frac{\pi}{2} - f_1 \left( \frac{\varepsilon}{\alpha} \right) \right) \alpha \beta \frac{N_1}{Wt} + B\sigma_2}^2 - 8A \left( C\sigma_2 + \frac{N_1}{2Wt} \right)^2
\]

(40)

3. Cases for “crack type III”

The valid region of “crack type III” meets \( \delta \in \left[ 0, \frac{1}{2} - \kappa - \alpha \right] \), the normalized equilibrium equation can be obtained.

\[
\begin{align*}
n_{nt} &= -\delta \left( S_i^* - S_i \right) + \left( 1 - \frac{\pi}{2} \alpha \beta \right) S_i^*
\\m_{nt} &= -2\delta(1-\delta) \left( S_i^* - S_i \right) - 2\pi\alpha\beta\kappa S_i^*
\end{align*}
\]

(41)

Using Eqns. (3), (4), (5), (10) and (14), Eqn. (42) can be obtained:

\[
A \left( 4\sigma_y^2 - 3\sigma_{2L}^2 \right) - \left( \frac{2M_{nt}}{Wt^2} - \pi\alpha\beta \frac{N_{nt}}{2Wt} - B\sigma_{2L} \right) \sqrt{4\sigma_y^2 - 3\sigma_{2L}^2} + 2 \left( C\sigma_{2L} + \frac{N_{nt}}{2Wt} \right)^2 = 0
\]

(42)

where, the coefficients \( A, B, C \) in Eqn.(42) are the same as those defined by Eqn.(33).

Similarly, the solution can be gotten as follow.

\[
\sigma_{ref}^2 = \frac{3}{4} \sigma_y^2 + \left[ \left( \frac{2M_1}{Wt^2} - \pi\alpha\beta \frac{N_1}{2Wt} - B\sigma_2 \right) + \sqrt{\left( \frac{2M_1}{Wt^2} - \pi\alpha\beta \frac{N_1}{2Wt} - B\sigma_2 \right)^2 - 8A \left( C\sigma_2 + \frac{N_1}{2Wt} \right)^2} \right]^2
\]

(43)

### 2.4 Summary of solutions

The solutions of “crack type I” in stress distribution A and stress distribution B can be unified as follow:

\[
\sigma_{ref}^2 = \frac{3}{4} \sigma_y^2 + \left[ \frac{2M_1}{Wt^2} + \pi\alpha\beta \frac{N_1}{2Wt} + B\sigma_2 \right] + \sqrt{\left( \frac{2M_1}{Wt^2} + \pi\alpha\beta \frac{N_1}{2Wt} + B\sigma_2 \right)^2 - 8A \left( C\sigma_2 + \frac{N_1}{2Wt} \right)^2} \right]^2
\]

(44)
Using Eqns.(3), (4), (6) and (44), Eqn.(45) can be gotten.

\[
\sigma^2_{ref} = \sigma^* + \frac{3}{4} \lambda^2 + \left[ \frac{4\lambda_1 + \pi \alpha \beta + B \lambda_2}{4A} \right]^2
\]

where, the coefficients A, B, C are defined by Eqn.(13).

The solutions of "crack type II" in stress distribution A and stress distribution B can be unified as follow:

\[
\sigma^2_{ref} = \frac{3}{4} \lambda^2 + \left[ \frac{2M_1}{W^2} \left( \frac{\pi}{2} - f_1 \left( \frac{e}{\alpha} \right) \right) \alpha \beta + B \lambda_2 \right] + \left[ \frac{2M_1}{W^2} \left( \frac{\pi}{2} - f_1 \left( \frac{e}{\alpha} \right) \right) \alpha \beta + B \lambda_2 \right] - 8A \left( C \lambda_2 + 1 \right)^2
\]

Using Eqns.(3), (4), (6) and (46), Eqn.(47) can be gotten.

\[
\sigma^2_{ref} = \frac{3}{4} \lambda^2 + \left[ \frac{4\lambda_1 + 2 \left( \frac{\pi}{2} - f_1 \left( \frac{e}{\alpha} \right) \right) \alpha \beta + B \lambda_2}{4A} \right]^2
\]

where, the coefficients A, B, C are defined by Eqn.(28).

The solutions of "crack type III" in stress distribution A and stress distribution B can be unified as follow:

\[
\sigma^2_{ref} = \frac{3}{4} \lambda^2 + \left[ \frac{2M_1}{W^2} - \pi \alpha \beta + B \lambda_2 \right] + \left[ \frac{2M_1}{W^2} - \pi \alpha \beta + B \lambda_2 \right] - 8A \left( C \lambda_2 + 1 \right)^2
\]

Using Eqns.(3), (4), (6) and (48), Eqn.(49) can be gotten.

\[
\sigma^2_{ref} = \frac{3}{4} \lambda^2 + \left[ \frac{4\lambda_1 - \pi \alpha \beta - B \lambda_2}{4A} \right]^2
\]

where, the coefficients A, B, C are defined by Eqn.(33).

### 2.5 Definition of the valid regions

Just as reference[2] described, reference stress solutions are chosen based on the position of neutral axis \(\delta\). But \(\delta\) is an unanswered value and unknown before the solution can be obtained. In the given geometric parameters \(\alpha, \beta, \kappa\) and load variables \(M_1, N_1, \sigma_2\), valid region of reference stress solutions can be expressed by \(\lambda_1\) for proportional loading (known \(\lambda_1\) and \(\lambda_2\)). The load ratio \(\lambda_1\) is defined by \(\lambda_{11}, \lambda_{12}, \lambda_{13}\) and \(\lambda_{14}\) corresponding to \(\delta = 1, \delta = \frac{1}{2} - \kappa + \alpha, \delta = \frac{1}{2} - \kappa - \alpha\) and \(\delta = 0\). The values of \(\lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{14}\) can be expressed as follows:
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\[
\left\{ \begin{array}{l}
\lambda_1 = \lambda_4 = -\frac{\pi \alpha \beta \kappa}{2 - \pi \alpha \beta} \\
\lambda_2 = -\left[ \frac{1}{2} - 2(k - \alpha)^2 - \pi \alpha \beta \right] \psi - \frac{\pi}{4} \alpha \beta \lambda_2 \\
\lambda_3 = -\left[ \frac{1}{2} - 2(k + \alpha)^2 + \pi \alpha \beta \right] \psi - \frac{\pi}{4} \alpha \beta \lambda_2
\end{array} \right. \quad (50)
\]

where, \( \psi = 1 - \left(\frac{1}{2} - \frac{\pi}{4} \alpha \beta\right) \lambda_2 \), the effect of \( \psi \) defined the validity ranges has been discussed in reference[2].

(1) The solutions are as follows when \( N_1 \neq 0 \):

- If \( \lambda_1 \geq \lambda_3 \geq \lambda_1 \) (\( \lambda_1 - \lambda_3 > 0 \)) \( (\psi > 0) \)
- If \( \lambda_1 \geq \lambda_3 \geq \infty \) (\( \lambda_1 - \lambda_3 < 0 \)) \( (\psi > 0) \)
- Or \( \lambda_1 \geq \lambda_3 \geq \infty \) (\( \lambda_1 - \lambda_3 > 0 \)) \( (\psi < 0) \)

Eqns. (44) and (45) will be met and the coefficients are determined by Eqn. (13).

(2) The solutions are as follows when \( N_1 = 0 \):

\[
\left\{ \begin{array}{l}
\lambda_1 \geq \lambda_4 \geq \lambda_3, \quad (\psi > 0) \\
\lambda_1 \geq \lambda_4 \geq \lambda_3, \quad (\psi < 0)
\end{array} \right.
\]

Eqns. (46) and (47) will be met and the coefficients are determined by Eqn. (28).

\[
\left\{ \begin{array}{l}
\lambda_3 \geq \lambda_1 \geq \lambda_4, \quad (\psi < 0)
\end{array} \right.
\]

Eqns. (48) and (49) will be met and the coefficients are determined by Eqn. (33).

\[
\left\{ \begin{array}{l}
\lambda_3 \geq \lambda_1 \geq \lambda_4, \quad (\psi > 0)
\end{array} \right.
\]

Eqns. (50) will be met and the coefficients are determined by Eqn. (33).

\[
\left\{ \begin{array}{l}
\lambda_3 \geq \lambda_1 \geq \lambda_4, \quad (\psi < 0)
\end{array} \right.
\]

Four \( \lambda_i \) values corresponding to \( \delta = 1 \), \( \delta = \frac{1}{2} - \kappa + \alpha \), \( \delta = \frac{1}{2} - \kappa - \alpha \) and \( \delta = 0 \) can be gotten when \( N_1 = 0 \) as follows:

\[
\lambda_{31} = -\infty
\]

\[
\lambda_{33} = \frac{2}{(3 \pi \alpha \beta - 3)(\kappa - \alpha)} \left( \frac{3 \pi \alpha \beta - 3}{2} - \frac{3 \pi \alpha \kappa + \frac{3}{2} \pi \kappa + \frac{3}{8} \pi}{\alpha \beta} + \frac{3}{4} \right)
\]

\[
\lambda_{32} = \frac{2}{(3 \pi \alpha \beta - 3)(\kappa + \alpha)} \left( \frac{3 \pi \alpha \beta - 3}{2} - \frac{3 \pi \alpha \kappa + \frac{3}{2} \pi \kappa + \frac{3}{8} \pi}{\alpha \beta} + \frac{3}{4} \right)
\]

\[
\lambda_{34} = \infty
\]

For \( \lambda_{31} \leq \lambda_3 \leq \lambda_{32} \),

\[
\sigma_{pw}^2 = \frac{3}{4} \sigma_i^2 + \left[ \frac{2M}{Wt^2} + B \sigma_i + \sqrt{\left( \frac{2M}{Wt^2} + B \sigma_i \right)^2 - 8AC^2 \sigma_i^2} \right]^2
\]

\[
(52)
\]
or

\[
\sigma_{\text{ref}}^2 = \sigma_{\text{r}}^2 \left[ \frac{3}{4} \lambda_3^2 + \left( \frac{\frac{2}{3} + B \lambda_3}{\sqrt{\frac{2}{3} + B \lambda_3}} \right)^2 + \frac{8A(\lambda_3)^2}{4A} \right] ^2 \]  

(53)

where, the coefficients \( A, B, C \) in Eqns. (52) and (53) are the same as those defined by Eqn.(13).

For \( \lambda_{12} \leq \lambda_3 \leq \lambda_{33} \), Eqns. (52) and (53) meets the solution but the coefficients \( A, B, C \) are the same as those defined by Eqn.(28).

Note that the definition of " \( Z \) " in Eqn.(26) should be changed by

\[
Z = \frac{3\alpha\left(\frac{\pi}{2} \alpha \beta - 1\right) \lambda_3}{6\pi \alpha \beta \lambda_3 + 4}.
\]

For \( \lambda_{33} \leq \lambda_3 \leq \lambda_{34} \),

\[
\sigma_{\text{ref}}^2 = \frac{3}{4} \sigma_{\text{r}}^2 + \left( \frac{2M_1}{Wt^2} - B \sigma_2 \right)^2 + \frac{2M_1}{Wt^2} - B \sigma_2 \right)^2 - \frac{8A\sigma_2 + \frac{N_1}{2Wt}}{4A} \]  

(54)

or

\[
\sigma_{\text{ref}}^2 = \sigma_{\text{r}}^2 \left[ \frac{3}{4} \lambda_3^2 + \left( \frac{\frac{2}{3} - B \lambda_3}{\sqrt{\frac{2}{3} - B \lambda_3}} \right)^2 + \frac{8A(\lambda_3)^2}{4A} \right] ^2 \]  

(55)

where, the coefficients \( A, B, C \) in Eqns. (54) and (55) are the same as those defined by Eqn.(33).

### 2.6 Comparison of limit load solution

Load ratio \( \lambda_1 \) and \( \lambda_2 \) are constant, the relation of reference stress and limit load can be expressed as

\[
\sigma_{\text{ref}} = \frac{N_1}{N_{1L}(\lambda_1, \lambda_2, \sigma)} \sigma_y = \frac{M_1}{M_{1L}(\lambda_1, \lambda_2, \sigma)} \sigma_y = \frac{\sigma_2}{\sigma_{2L}(\lambda_1, \lambda_2, \sigma)} \sigma_y
\]

(56)

Using Eqns.(2), (6) and (50), the relation of reference stress and limit load can be gotten as

\[
\sigma_{\text{ref}} = L \sigma_y = \frac{\sigma_\text{m}}{n_{1L}} = \frac{2\sigma_\text{n}}{3m_{1L}}
\]

(57)

where, \( n_{1L} = \frac{\sigma_\text{m}}{\sigma_{\text{ref}}} \) (when \( N_1 \neq 0 \)), \( m_{1L} = \frac{2\sigma_\text{n}}{3\sigma_{\text{ref}}} \) (when \( N_1 = 0 \)). Note that plus or minus \( n_{1L} \) and \( m_{1L} \) depends on their loading directions.

Actual embedded defect is modelled by a circumscribing ellipse or rectangular. In order to solve conveniently, rectangular cracks are often used instead of elliptical cracks in engineering practice. However, the load carrying capacity of component is underestimated. Therefore, the evaluation may be conservative. Comparison of limit load solution between plates with rectangular and elliptical crack under the same geometric parameters and loading conditions has been done by \( n_{1L} - m_{1L} \) (\( N_1 \neq 0 \)) and \( n_{2L} - m_{1L} \) (\( N_1 = 0 \)) space, shown in Fig.2. The geometric...
parameters and loading conditions of plate are the same \((\alpha = 0.25, \beta = 0.25, \kappa = 0.1, a/c = 0.3, \lambda_2 = 1)\). It can be seen that limit load solution between plates with rectangular crack is slight conservative compared to plates with elliptical crack.

\[ J = J_e \left[ \frac{E \varepsilon_{ref}}{L_e \sigma_0} + L_e \frac{\sigma_0}{2E \varepsilon_{ref}} \right] \]  \hspace{1cm} (58)

where, \(\varepsilon_{ref}\) is the reference strain corresponding to the reference stress, \(\sigma_0\) is the yield stress or 0.2% proof stress. \(L_e\) is a parameter used to measure the load close to the structural plastic instability. \(E\) is the Young’s modulus \([16, 17]\).

\(J_e\) may be obtained from stress intensity factor, \(K_I\), from Eqn.\((59)\):

\[ J_e = \left[ K_I \right]^2 \left( 1 - \nu^2 \right) \]  \hspace{1cm} (59)

where, \(\nu\) is Poisson’s ratio.

### 3.2 Finite element analysis

\(K_I\) and \(J\) are evaluated by FE analyses for a plate with embedded elliptical crack under combined biaxial forces and cross-thickness bending. A typical model has been modeled in Fig.3(a) with the conditions of geometric parameters \((\alpha = 0.25, \beta = 0.25, a/c = 0.3, \kappa = 0.1)\) and load cases \((\lambda_1 = 0, 0.2, \infty\) and \(\lambda_2 = -1, 0, 1)\). The material properties used are as follows: \(E = 200000\) and \(\nu = 0.3\) for elastic analysis, \(E = 200000\), \(\nu = 0.3\), \(\alpha = 1\), \(n = 5\) and \(\sigma_y = 200\) for elastic–plastic analyses with Ramberg-Osgood material.

The software of ABAQUS\([13]\) is used to build model with 20-node isoparametric quadratic solid elements with reduced integration. \(J\) values are calculated by the in-built contour integrator function at 11 positions defined by \(\theta\) along the crack front, shown in Fig.3(b). \(J\) is evaluated on 15 contours around the crack tip. The maximum difference between obtained \(J\) values and average is less than 3%. Note that \(J\) values obtained at the closed crack tip are excluded.
Stress intensity factor, $K_i$, may be expressed as

$$K_i = \sigma_0 \sqrt{\pi a} f_{com}(\alpha, \beta, \kappa, a/c, \lambda)$$  \hspace{1cm} (60)

where, according to the definition by Lei[14], $f_{com}$ is a geometry and load type dependent function, it meets $f_{com} = f_t + 6\lambda f_b$ ($f_t$ is the geometry function for remote tension and $f_b$ for bending load); According to the definition of EPRI[15],

$$a_c = a + \varphi r, \quad r = \frac{1}{1 + (N_1 / N_{1L})^2}$$

when plane strain, $\varphi = \frac{1}{6\pi n + 1} \left( \frac{K_i}{\sigma_0} \right)^2$.

The values of the function $f_{com}$ obtained from the FE analysis for different $\lambda_i$ values has been plotted in Fig. 4. It can be seen that the predicted results are in good agreement with finite element analysis results.

Crack tip point in $\theta = -\pi / 2$ (shown in Fig.3(b)) is usually considered to be the most dangerous position. The $J$ values along the crack front of $\theta = -\pi / 2$ with different $\lambda_2$ are estimated by reference stress method and FE analysis, shown in Fig. 5--Fig. 7. Effect of stress parallel to the crack plane on $J$ values can be seen that, for the cases with $\sigma_1 \sigma_2 < 0$, $J$ values continue to become smaller with the increase of $\lambda_2$. So ignoring $\sigma_2$ may significantly overestimate $J$ values; for the cases with $\sigma_1 \sigma_2 > 0$, $J$ values increase slightly bigger at the beginning, then they become smaller after exceeding a certain $\lambda_2$ value, ignoring $\sigma_2$ may underestimate $J$ values. Predicted $J$ values gotten by reference stress method are always lower than FE analysis results. At the same $\lambda_2$ value, it can be reasonably estimated by reference stress method when the loading case is pure tension or small $\lambda_1$ value. But it exists significant errors when the loading case is pure bending and large load. Note that the strain of the plate extends outward from the edge of the crack, at the same time, it extends from the edge of the plate to interior. But only plastic correction at the crack tip is considered by the reference stress method. So the evaluation result may be unreasonable and dangerous when bending load is dominant.
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Fig. 5 Comparison of $J$ values gotten by reference stress method and FE results when $\lambda_1 = 0$

Fig. 6 Comparison of $J$ values gotten by reference stress method and FE results when $\lambda_1 = 0.2$

Fig. 7 Comparison of $J$ values gotten by reference stress method and FE results $\lambda_1 = \infty$
section iv

conclusion

In this paper, reference stress solutions for a plate with embedded off-set elliptical crack under combined biaxial tensile/compressive force/stress and cross-thickness bending moment have been gotten. The results show that the ignored compressive stress parallel to the crack plane will underestimate J integral of crack tip.

section v

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references


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